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Compound Determinants.

By William H. Metzler.

Introduction.

In the theory of compound determinants it has been shown that any minor of a compound determinant $\Delta_{(m)}$, formed from the minors of order m of a determinant Δ, can be expressed in terms of powers of Δ and the complementary of the corresponding minor of the reciprocal of $\Delta_{(m)}$. No one, as far as I have been able to ascertain, has expressed these minors of $\Delta_{(m)}$ directly in terms of the minors of various orders of Δ. I shall in this paper show how to express these minors in this way, and shall also show how to express the sums of the principal minors (of various orders) of the system of compound determinants formed from the minors of Δ in terms of the principal minors (of various orders) of Δ.

From these relations I shall deduce a theorem in regard to symmetric and skew symmetric determinants from which I shall in turn deduce a theorem in matrices. The paper will therefore be divided into two parts, the first relating to determinants and the second to matrices.

I shall first state some theorems which will be required, referring the reader for their proof, where proof is necessary, to the well-known texts by Baltzer, Gordan, Scott, Muir and others.

I.—Determinants.

§1.—*Auxiliary Theorems.*

1. The minors of order m formed from the constituents of a determinant of order ω, are λ^2 in number, where $\lambda = \binom{\omega}{m} = \omega_m = \frac{\omega(\omega-1)\ldots.(\omega-m+1)}{m!}$*,

* Baltzer, Theorie der Determinanten, §2, 4, §4, 1. Scott, Theory of Determinants, Chap. III, 1.

and may therefore be arranged in a square. Let us then form a compound determinant $\Delta_{(m)}$ from the minors of order m of a given determinant Δ of order ω, writing in the same row all those minors proceeding from the same selection of rows, and similarly for the columns; and also its reciprocal (adjugate) determinant $\Delta_{(\omega-m)}$ formed from the minors of order $\omega - m$, any constituent of the one being the complementary minor of the corresponding constituent in the other.

2. To express $\Delta_{(m)}$ and $\Delta_{(\omega-m)}$ in terms of Δ we have the relations

$$\Delta_{(m)}\Delta_{(\omega-m)} = \Delta^{\lambda} = \Delta^{\omega_m},$$

$$\Delta_{(m)} = \Delta^{(\omega-1)_{m-1}}$$

and

$$\Delta_{(\omega-m)} = \Delta^{(\omega-1)_m} = \Delta^{(\omega-1)_{\omega-m-1}}.*$$

If $m = \omega - m$, then $\Delta_{(m)} = \Delta_{(\omega-m)} = \pm \Delta^{\frac{\lambda}{2}}$.

3. If Δ is a symmetric determinant, then it is quite obvious that $\Delta_{(m)}$ and $\Delta_{(\omega-m)}$ are symmetric. If Δ is skew symmetric, then $\Delta_{(i)}$ will be either symmetric or skew symmetric according as i is even or odd.

4. The reciprocal of every constituent of the compound determinant $\Delta_{(m)}$ is a minor of order m of the compound determinant $\Delta_{(m-1)}$. The converse, viz. that every minor of order m of $\Delta_{(m-1)}$ is the reciprocal of some constituent of $\Delta_{(m)}$ is, however, not true.†

5. If a determinant vanishes, then all minors of order two of its reciprocal vanish.

§2.—*Expressions for Sums of Principal Minors of Compound Determinants.*

To illustrate the method employed in obtaining the expression for any minor of a compound determinant in terms of minors of Δ, let us consider a few particular cases. After considering more or less in detail these particular cases, we shall proceed to the general case of a determinant Δ of order ω, and express the sum of the principal minors of order two, of each of the system of compound determinants formed from Δ, directly in terms of minors of Δ.

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* Baltzer, §7, 6. Scott, Chap. V, 9.

† Cf. Scott, Chap. V, 11.

A.—*Determinant of order four.*

6. Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix},$$

then

$$\Delta_{(2)} = \begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\23}} & A_{\substack{12\\24}} & A_{\substack{12\\34}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} & A_{\substack{13\\23}} & A_{\substack{13\\24}} & A_{\substack{13\\34}} \\ A_{\substack{14\\12}} & A_{\substack{14\\13}} & A_{\substack{14\\14}} & A_{\substack{14\\23}} & A_{\substack{14\\24}} & A_{\substack{14\\34}} \\ A_{\substack{23\\12}} & A_{\substack{23\\13}} & A_{\substack{23\\14}} & A_{\substack{23\\23}} & A_{\substack{23\\24}} & A_{\substack{23\\34}} \\ A_{\substack{24\\12}} & A_{\substack{24\\13}} & A_{\substack{24\\14}} & A_{\substack{24\\23}} & A_{\substack{24\\24}} & A_{\substack{24\\34}} \\ A_{\substack{34\\12}} & A_{\substack{34\\13}} & A_{\substack{34\\14}} & A_{\substack{34\\23}} & A_{\substack{34\\24}} & A_{\substack{34\\34}} \end{vmatrix}$$

where $A_{\substack{rt\\su}}$ denotes the minor complementary to $(a_{rs}a_{tu})$ with the proper sign attached, and where $A_{\substack{rt\\su}} = -A_{\substack{rt\\us}}$,* and

$$\Delta_{(3)} = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix},$$

where A_{rs} denotes the minor complementaty to a_{rs} with the proper sign, so that

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14} = \Delta,$$
$$\text{etc.,} \qquad\qquad \text{etc.}$$

For the reciprocal of $\Delta_{(2)}$ which will be designated by $\Delta'_{(2)}$ we have

$$\Delta'_{(2)} = \begin{vmatrix} A_{\substack{34\\34}} & A_{\substack{34\\42}} & A_{\substack{34\\23}} & A_{\substack{34\\14}} & A_{\substack{34\\31}} & A_{\substack{34\\12}} \\ A_{\substack{24\\43}} & A_{\substack{24\\24}} & A_{\substack{24\\32}} & A_{\substack{24\\41}} & A_{\substack{24\\13}} & A_{\substack{24\\21}} \\ A_{\substack{23\\34}} & A_{\substack{23\\42}} & A_{\substack{23\\23}} & A_{\substack{23\\14}} & A_{\substack{23\\31}} & A_{\substack{23\\12}} \\ A_{\substack{14\\34}} & A_{\substack{14\\43}} & A_{\substack{14\\23}} & A_{\substack{14\\14}} & A_{\substack{14\\31}} & A_{\substack{14\\12}} \\ A_{\substack{13\\43}} & A_{\substack{13\\24}} & A_{\substack{13\\32}} & A_{\substack{13\\41}} & A_{\substack{13\\13}} & A_{\substack{13\\21}} \\ A_{\substack{12\\34}} & A_{\substack{12\\42}} & A_{\substack{12\\23}} & A_{\substack{12\\14}} & A_{\substack{12\\31}} & A_{\substack{12\\12}} \end{vmatrix}$$

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* *Vide* Gordan, Determinanten, §4, 42–49.

so that

$$\underset{12}{A_{12}}\underset{34}{A_{34}}+\underset{13}{A_{12}}\underset{42}{A_{34}}+\underset{14}{A_{12}}\underset{23}{A_{34}}+\underset{23}{A_{12}}\underset{14}{A_{34}}+\underset{24}{A_{12}}\underset{31}{A_{34}}+\underset{34}{A_{12}}\underset{12}{A_{34}}=\Delta,$$

etc., etc.

7. We obtain by the ordinary methods expressions for the minors of any of these compound determinants in terms of powers of Δ and the complementaries of the corresponding minors of their reciprocals. For the principal minors and their sums we have,

in case of minors of $\Delta_{(2)}$,

$$(\underset{12}{A_{12}}\,\underset{13}{A_{13}}\,\underset{14}{A_{14}}\,\underset{23}{A_{23}}\,\underset{24}{A_{24}})=\Delta^2\underset{12}{A_{12}},$$

and $\therefore$

$$\Sigma\,(\underset{12}{A_{12}}\,\underset{13}{A_{13}}\,\underset{14}{A_{14}}\,\underset{23}{A_{23}}\,\underset{24}{A_{24}})=\Delta^2\Sigma\underset{12}{A_{12}},$$

$$(\underset{12}{A_{12}}\,\underset{13}{A_{13}}\,\underset{14}{A_{14}}\,\underset{23}{A_{23}})=\Delta\,(\underset{12}{A_{12}}\,\underset{13}{A_{13}}),$$

and $\therefore$

$$\Sigma\,(\underset{12}{A_{12}}\,\underset{13}{A_{13}}\,\underset{14}{A_{14}}\,\underset{23}{A_{23}})=\Delta\Sigma\,(\underset{12}{A_{12}}\,\underset{13}{A_{13}});$$

in case of minors of $\Delta_{(3)}$,

$$(A_{11}\,A_{22}\,A_{33})=\Delta^2 a_{44},$$

and $\therefore$

$$\Sigma\,(A_{11}\,A_{22}\,A_{33})=\Delta^2\Sigma a_{11},$$

$$(A_{11}\,A_{22})=\Delta\underset{12}{A_{12}},$$

and $\therefore$

$$\Sigma\,(A_{11}\,A_{22})=\Delta\Sigma\underset{12}{A_{12}}.$$

8. In the case of minors of $\Delta_{(2)}$ of the third order, we get nothing simpler in this way. We may, however, obtain simple results as follows:

$$\begin{vmatrix}\underset{12}{A_{12}} & \underset{13}{A_{12}} & \underset{14}{A_{12}}\\ \underset{12}{A_{13}} & \underset{13}{A_{13}} & \underset{14}{A_{13}}\\ \underset{12}{A_{14}} & \underset{13}{A_{14}} & \underset{14}{A_{14}}\end{vmatrix}\cdot\begin{vmatrix}a_{22} & a_{23} & a_{24}\\ a_{32} & a_{33} & a_{34}\\ a_{42} & a_{43} & a_{44}\end{vmatrix}=\begin{vmatrix}A_{11} & 0 & 0\\ 0 & A_{11} & 0\\ 0 & 0 & A_{11}\end{vmatrix}$$

$$=A_{11}^3,$$

$$\therefore\quad(\underset{12}{A_{12}}\,\underset{13}{A_{13}}\,\underset{14}{A_{14}})=A_{11}^2.$$

Similarly,

$$(\underset{12}{A_{12}}\,\underset{23}{A_{23}}\,\underset{24}{A_{24}})=A_{22}^2,$$

$$(\underset{13}{A_{13}}\,\underset{23}{A_{23}}\,\underset{34}{A_{34}})=A_{33}^2,$$

$$(\underset{14}{A_{14}}\,\underset{24}{A_{24}}\,\underset{34}{A_{34}})=A_{44}^2.$$

The product

$$\begin{vmatrix} \underset{12}{A_{12}} & \underset{13}{A_{12}} & \underset{14}{A_{12}} & \underset{23}{A_{12}} & \underset{24}{A_{12}} \\ \underset{12}{A_{13}} & \underset{13}{A_{13}} & \underset{14}{A_{13}} & \underset{23}{A_{13}} & \underset{24}{A_{13}} \\ 0 & 0 & 1 & 0 & 0 \\ \underset{12}{A_{23}} & \underset{13}{A_{23}} & \underset{14}{A_{23}} & \underset{23}{A_{23}} & \underset{24}{A_{23}} \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{32} & a_{33} & a_{34} & 0 & 0 \\ a_{42} & a_{43} & a_{44} & 0 & 0 \\ -a_{31}^{*} & 0 & 0 & a_{33} & a_{34} \\ -a_{41}^{*} & 0 & 0 & a_{43} & a_{44} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & A_{11} & 0 & A_{12} & 0 \\ a_{24} & a_{34} & a_{44} & 0 & 0 \\ -A_{31} & A_{21} & 0 & A_{22} & 0 \\ 0 & 0 & 0 & a_{34} & a_{44} \end{vmatrix}$$

$$= a_{44}^2 A_{11} (A_{11} A_{22} - A_{12} A_{21})$$

$$= a_{44}^2 \Delta A_{11} \underset{12}{A_{12}},$$

$$\therefore \quad (\underset{12}{A_{12}} \underset{13}{A_{13}} \underset{23}{A_{23}}) = a_{44}^2 \Delta.$$

Similarly,

$$(\underset{12}{A_{12}} \underset{14}{A_{14}} \underset{24}{A_{24}}) = a_{33}^2 \Delta,$$

$$(\underset{13}{A_{13}} \underset{14}{A_{14}} \underset{34}{A_{34}}) = a_{22}^2 \Delta,$$

$$(\underset{23}{A_{23}} \underset{24}{A_{24}} \underset{34}{A_{34}}) = a_{11}^2 \Delta.$$

The product

$$\begin{vmatrix} \underset{12}{A_{12}} & \underset{13}{A_{12}} & \underset{14}{A_{12}} & \underset{23}{A_{12}} & \underset{24}{A_{12}} \\ \underset{12}{A_{13}} & \underset{13}{A_{13}} & \underset{14}{A_{13}} & \underset{23}{A_{13}} & \underset{24}{A_{13}} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \underset{12}{A_{24}} & \underset{13}{A_{24}} & \underset{14}{A_{24}} & \underset{23}{A_{24}} & \underset{24}{A_{24}} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{32} & a_{33} & a_{34} & 0 & 0 \\ a_{42} & a_{43} & a_{44} & 0 & 0 \\ -a_{31} & 0 & 0 & a_{33} & a_{34} \\ -a_{41} & 0 & 0 & a_{43} & a_{44} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & A_{11} & 0 & A_{12} & 0 \\ a_{24} & a_{34} & a_{44} & 0 & 0 \\ 0 & 0 & 0 & a_{33} & a_{43} \\ -A_{41} & 0 & A_{21} & 0 & A_{22} \end{vmatrix}$$

$$= A_{11} \{ A_{11} a_{44} a_{33} A_{22} - a_{34} a_{43} A_{12} A_{21} \}$$

$$= A_{11} \underset{12}{A_{12}} \{ A_{11} A_{22} + \Delta a_{33} a_{44} - \Delta \underset{12}{A_{12}} \},$$

$$\therefore \quad (\underset{12}{A_{12}} \underset{13}{A_{13}} \underset{24}{A_{24}}) = a_{33} a_{44} \Delta + A_{11} A_{22} - \Delta \underset{12}{A_{12}}.$$

---

* The reason for writing these constituents negative is obvious.

Similarly,

$$
\begin{aligned}
(A_{\substack{12\\12}} A_{\substack{13\\13}} A_{\substack{34\\34}}) &= a_{22}\, a_{44}\, \Delta + A_{11}\, A_{33} - \Delta A_{\substack{13\\13}}\,, \\
(A_{\substack{12\\12}} A_{\substack{14\\14}} A_{\substack{23\\23}}) &= a_{33}\, a_{44}\, \Delta + A_{11}\, A_{22} - \Delta A_{\substack{12\\12}}\,, \\
(A_{\substack{12\\12}} A_{\substack{14\\14}} A_{\substack{34\\34}}) &= a_{22}\, a_{33}\, \Delta + A_{11}\, A_{44} - \Delta A_{\substack{14\\14}}\,, \\
(A_{\substack{13\\13}} A_{\substack{14\\14}} A_{\substack{23\\23}}) &= a_{22}\, a_{44}\, \Delta + A_{11}\, A_{33} - \Delta A_{\substack{13\\13}}\,, \\
(A_{\substack{13\\13}} A_{\substack{14\\14}} A_{\substack{24\\24}}) &= a_{22}\, a_{33}\, \Delta + A_{11}\, A_{44} - \Delta A_{\substack{14\\14}}\,, \\
(A_{\substack{12\\12}} A_{\substack{23\\23}} A_{\substack{34\\34}}) &= a_{11}\, a_{44}\, \Delta + A_{22}\, A_{33} - \Delta A_{\substack{23\\23}}\,, \\
(A_{\substack{12\\12}} A_{\substack{24\\24}} A_{\substack{34\\34}}) &= a_{11}\, a_{33}\, \Delta + A_{22}\, A_{44} - \Delta A_{\substack{24\\24}}\,, \\
(A_{\substack{13\\13}} A_{\substack{23\\23}} A_{\substack{24\\24}}) &= a_{11}\, a_{44}\, \Delta + A_{22}\, A_{33} - \Delta A_{\substack{23\\23}}\,, \\
(A_{\substack{13\\13}} A_{\substack{24\\24}} A_{\substack{34\\34}}) &= a_{11}\, a_{22}\, \Delta + A_{33}\, A_{44} - \Delta A_{\substack{34\\34}}\,, \\
(A_{\substack{14\\14}} A_{\substack{23\\23}} A_{\substack{24\\24}}) &= a_{11}\, a_{33}\, \Delta + A_{22}\, A_{44} - \Delta A_{\substack{24\\24}}\,, \\
(A_{\substack{14\\14}} A_{\substack{23\\23}} A_{\substack{34\\34}}) &= a_{11}\, a_{22}\, \Delta + A_{33}\, A_{44} - \Delta A_{\substack{34\\34}}\,.
\end{aligned}
$$

Taking the sum of all principal minors of order three we get

$$\Sigma\,(A_{\substack{12\\12}} A_{\substack{13\\13}} A_{\substack{14\\14}}) = (\Sigma A_{11})^2 + \Delta\left\{(\Sigma a_{11})^2 - 2\Sigma A_{\substack{12\\12}}\right\}.$$

It is obvious that the expression for any minor may be found in this way, but being particularly interested in the expressions for principal minors, and especially in principal minors of order two, the examples will be almost entirely confined to those cases.

9. We shall now express the principal second minors of $\Delta_{(2)}$ directly in terms of minors of Δ.

The product

$$
\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ a_{24} & a_{34} & a_{44} \end{vmatrix}
$$

$$= a_{44}\, A_{11}^2\,,$$

$$\therefore \quad (A_{\substack{12\\12}} A_{\substack{13\\13}}) = a_{44}\, A_{11}\,.$$

Similarly,

$$
\begin{aligned}
(A_{\substack{12\\12}}\, A_{\substack{14\\14}}) &= a_{33}\, A_{11}\,,\\
(A_{\substack{12\\12}}\, A_{\substack{23\\23}}) &= a_{44}\, A_{22}\,,\\
(A_{\substack{12\\12}}\, A_{\substack{24\\24}}) &= a_{33}\, A_{22}\,,\\
(A_{\substack{13\\13}}\, A_{\substack{14\\14}}) &= a_{22}\, A_{11}\,,\\
(\mathrm{A}_{\substack{13\\13}}\, A_{\substack{23\\23}}) &= a_{44}\, A_{33}\,,\\
(A_{\substack{13\\13}}\, A_{\substack{34\\34}}) &= a_{22}\, A_{33}\,,\\
(A_{\substack{14\\14}}\, A_{\substack{24\\24}}) &= a_{33}\, A_{44}\,,\\
(A_{\substack{14\\14}}\, A_{\substack{34\\34}}) &= a_{22}\, A_{44}\,,\\
(A_{\substack{23\\23}}\, A_{\substack{24\\24}}) &= a_{11}\, A_{22}\,,\\
(A_{\substack{23\\23}}\, A_{\substack{34\\34}}) &= a_{11}\, A_{33}\,,\\
(A_{\substack{24\\24}}\, A_{\substack{34\\34}}) &= a_{11}\, A_{44}\,.
\end{aligned}
$$

The product

$$
\begin{vmatrix}
A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\23}} & A_{\substack{12\\34}}\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0\\
A_{\substack{34\\12}} & A_{\substack{34\\13}} & A_{\substack{34\\14}} & A_{\substack{34\\23}} & A_{\substack{34\\34}}
\end{vmatrix}
\cdot
\begin{vmatrix}
a_{22} & a_{23} & a_{24} & 0 & 0\\
a_{32} & a_{33} & a_{34} & 0 & 0\\
a_{42} & a_{43} & a_{44} & 0 & 0\\
0 & a_{21} & 0 & a_{22} & -a_{24}\\
0 & a_{41} & 0 & a_{42} & -a_{44}
\end{vmatrix}
$$

$$
\begin{aligned}
&= \begin{vmatrix}
A_{11} & 0 & 0 & -A_{13} & 0\\
a_{23} & a_{33} & a_{43} & a_{21} & a_{41}\\
a_{24} & a_{34} & a_{44} & 0 & 0\\
0 & 0 & 0 & a_{22} & a_{42}\\
0 & -A_{41} & A_{31} & 0 & -A_{33}
\end{vmatrix}\\
&= A_{11}\{A_{\substack{13\\34}}(a_{34}\, A_{31} + a_{44}\, A_{41}) - a_{22}\, A_{33}\, A_{\substack{12\\12}}\}\\
&\quad + A_{13}\, a_{42}\{A_{41}\, A_{\substack{13\\12}} - A_{31}\, A_{\substack{14\\12}}\},
\end{aligned}
$$

which, after easy transformations,

$$
= A_{11}\, A_{\substack{13\\13}}\{\Delta - a_{11}\, A_{11} - a_{12}\, A_{12} - a_{21}\, A_{21} - a_{22}\, A_{22}\},
$$

$$
\therefore\quad (A_{\substack{12\\12}}\, A_{\substack{34\\34}}) = a_{11}\, A_{11} + a_{12}\, A_{12} + a_{21}\, A_{21} + a_{22}\, A_{22} - \Delta.
$$

Similarly,

$$(\underset{13}{A_{13}}\,\underset{24}{A_{24}}) = a_{11}A_{11} + a_{13}A_{13} + a_{31}A_{31} + a_{33}A_{33} - \Delta,$$

$$(\underset{14}{A_{14}}\,\underset{23}{A_{23}}) = a_{11}A_{11} + a_{14}A_{14} + a_{41}A_{41} + a_{44}A_{44} - \Delta,$$

$$\therefore\quad (\underset{12}{A_{12}}\,\underset{34}{A_{34}}) + (\underset{13}{A_{13}}\,\underset{24}{A_{24}}) + (\underset{14}{A_{14}}\,\underset{23}{A_{23}}) = a_{11}A_{11} + a_{22}A_{22} + a_{33}A_{33} + a_{44}A_{44} - \Delta.$$

Taking now the sum of all principal minors of order two, we get

$$\Sigma\,(\underset{12}{A_{12}}\,\underset{13}{A_{13}}) = (\Sigma a_{11})(\Sigma A_{11}) - \Delta.$$

10. Having the sum of the principal minors of any order, we have of course the sum of their complementaries. That is

$$\Sigma\,(\underset{12}{A_{12}}\,\underset{13}{A_{13}}\,\underset{14}{A_{14}}\,\underset{23}{A_{23}}) = \Delta\Sigma\,(\underset{12}{A_{12}}\,\underset{13}{A_{13}}) = \Delta\{(\Sigma a_{11})(\Sigma A_{11}) - \Delta\},$$

which is the expression for the sum of the minors of order four.

B.—*Determinant of order five.*

11. Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{48} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

then

$$\Delta_{(2)} = \begin{vmatrix} \underset{123}{A_{123}} & \underset{124}{A_{123}} & \underset{125}{A_{123}} & \underset{134}{A_{123}} & \underset{135}{A_{123}} & \underset{145}{A_{123}} & \underset{234}{A_{123}} & \underset{235}{A_{123}} & \underset{245}{A_{123}} & \underset{345}{A_{123}} \\ \underset{123}{A_{124}} & \underset{124}{A_{124}} & \underset{125}{A_{124}} & \underset{134}{A_{124}} & \underset{135}{A_{124}} & \underset{145}{A_{124}} & \underset{234}{A_{124}} & \underset{235}{A_{124}} & \underset{245}{A_{124}} & \underset{345}{A_{124}} \\ \underset{123}{A_{125}} & \underset{124}{A_{125}} & \underset{125}{A_{125}} & \underset{134}{A_{125}} & \underset{135}{A_{125}} & \underset{145}{A_{125}} & \underset{234}{A_{125}} & \underset{235}{A_{125}} & \underset{245}{A_{125}} & \underset{345}{A_{125}} \\ \underset{123}{A_{134}} & \underset{124}{A_{134}} & \underset{125}{A_{134}} & \underset{134}{A_{134}} & \underset{135}{A_{134}} & \underset{145}{A_{134}} & \underset{234}{A_{134}} & \underset{235}{A_{134}} & \underset{245}{A_{134}} & \underset{345}{A_{134}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{123}{A_{345}} & \underset{124}{A_{345}} & \underset{125}{A_{345}} & \underset{134}{A_{345}} & \underset{135}{A_{345}} & \underset{145}{A_{345}} & \underset{234}{A_{345}} & \underset{235}{A_{345}} & \underset{245}{A_{345}} & \underset{345}{A_{345}} \end{vmatrix},$$

$$\Delta_{(3)} = \left| \begin{array}{cccccccccc}
A_{\substack{45\\45}} & A_{\substack{45\\53}} & A_{\substack{45\\34}} & A_{\substack{45\\52}} & A_{\substack{45\\24}} & A_{\substack{45\\23}} & A_{\substack{45\\13}} & A_{\substack{45\\41}} & A_{\substack{45\\31}} & A_{\substack{45\\12}} \\
A_{\substack{35\\54}} & A_{\substack{35\\35}} & A_{\substack{35\\43}} & A_{\substack{35\\25}} & A_{\substack{35\\42}} & A_{\substack{35\\32}} & A_{\substack{35\\15}} & A_{\substack{35\\41}} & A_{\substack{35\\13}} & A_{\substack{35\\21}} \\
A_{\substack{34\\45}} & A_{\substack{34\\53}} & A_{\substack{34\\34}} & A_{\substack{34\\25}} & A_{\substack{34\\42}} & A_{\substack{34\\23}} & A_{\substack{34\\51}} & A_{\substack{34\\14}} & A_{\substack{34\\31}} & A_{\substack{34\\12}} \\
A_{\substack{25\\45}} & \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & A_{\substack{25\\12}} \\
A_{\substack{24\\54}} & \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & A_{\substack{24\\21}} \\
A_{\substack{23\\45}} & \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & A_{\substack{33\\12}} \\
A_{\substack{15\\54}} & \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & A_{\substack{15\\21}} \\
A_{\substack{14\\45}} & \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & A_{\substack{14\\12}} \\
A_{\substack{13\\54}} & \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & A_{\substack{13\\21}} \\
A_{\substack{12\\45}} & \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & A_{\substack{12\\12}}
\end{array} \right|$$

and

$$\Delta_{(4)} = \left| \begin{array}{ccccc}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\
A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\
A_{51} & A_{52} & A_{53} & A_{54} & A_{55}
\end{array} \right| .$$

12. Expressing the principal minors of these compound determinants in terms of powers of Δ and the complementaries of the corresponding minors of their reciprocals, we have for $\Delta_{(2)}$ and $\Delta_{(3)}$

$$(A_{\substack{123\\123}} A_{\substack{124\\124}} \ldots . A_{\substack{245\\245}}) = \Delta^3 A_{\substack{12\\12}},$$

and therefore

$$\Sigma\, (A_{\substack{123\\123}} A_{\substack{124\\124}} \ldots . A_{\substack{245\\245}}) = \Delta^3 \Sigma A_{\substack{12\\12}}.$$

Similarly,

$$\begin{aligned}
\Sigma\, (A_{\substack{123\\123}} A_{\substack{124\\124}} \ldots . A_{\substack{235\\235}}) &= \Delta^2 \Sigma\, (A_{\substack{12\\12}} A_{\substack{13\\13}}), \\
\Sigma\, (A_{\substack{123\\123}} A_{\substack{124\\124}} \ldots . A_{\substack{234\\234}}) &= \Delta \Sigma\, (A_{\substack{12\\12}} A_{\substack{13\\13}} A_{\substack{14\\14}}), \\
\Sigma\, (A_{\substack{123\\123}} A_{\substack{124\\124}} \ldots . A_{\substack{145\\145}}) &= \Sigma\, (A_{\substack{12\\12}} A_{\substack{13\\13}} A_{\substack{14\\14}} A_{\substack{15\\15}}), \\
\Delta \Sigma\, (A_{\substack{123\\123}} A_{\substack{124\\124}} \ldots . A_{\substack{135\\135}}) &= \Sigma\, (A_{\substack{12\\12}} A_{\substack{13\\13}} \ldots . A_{\substack{23\\23}}), \\
\Delta^2 \Sigma\, (A_{\substack{123\\123}} A_{\substack{124\\124}} \ldots . A_{\substack{134\\134}}) &= \Sigma\, (A_{\substack{12\\12}} A_{\substack{13\\13}} \ldots . A_{\substack{24\\24}}), \\
\Delta^3 \Sigma\, (A_{\substack{123\\123}} A_{\substack{124\\124}} A_{\substack{125\\125}}) &= \Sigma\, (A_{\substack{12\\12}} A_{\substack{13\\13}} \ldots . A_{\substack{25\\25}}), \\
\Delta^4 \Sigma\, (A_{\substack{123\\123}} A_{\substack{124\\124}}) &= \Sigma\, (A_{\substack{12\\12}} \ldots . A_{\substack{34\\34}});
\end{aligned}$$

19

and for $\Delta_{(4)}$,

$$(A_{11}\, A_{22}\, A_{33}\, A_{44}) = a_{55}\, \Delta^3,$$

and therefore

$$\Sigma\,(A_{11}\, A_{22}\, A_{33}\, A_{44}) = \Delta^3 \Sigma a_{11}.$$

Similarly,

$$\Sigma\,(A_{11}\, A_{22}\, A_{33}) = \Delta^2 \Sigma A_{\substack{123\\123}},$$

$$\Sigma\,(A_{11}\, A_{22}) = \Delta \Sigma A_{\substack{12\\12}}.$$

13. To obtain the expressions for principal minors of $\Delta_{(2)}$ of order two, we have the product

$$\begin{vmatrix} A_{\substack{123\\123}} & A_{\substack{123\\124}} & A_{\substack{123\\125}} \\ A_{\substack{124\\123}} & A_{\substack{124\\124}} & A_{\substack{124\\125}} \\ 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix} = \begin{vmatrix} A_{\substack{12\\12}} & 0 & 0 \\ 0 & A_{\substack{12\\12}} & 0 \\ a_{35} & a_{45} & a_{55} \end{vmatrix}$$

$$= a_{55}\, A^2_{\substack{12\\12}};$$

$$\therefore \quad (A_{\substack{123\\123}}\, A_{\substack{124\\124}}) = a_{55}\, A_{\substack{12\\12}}.$$

Similarly,

$$(A_{\substack{123\\123}}\, A_{\substack{125\\125}}) = a_{44}\, A_{\substack{12\\12}},$$

etc., etc.

The product

$$\begin{vmatrix} A_{\substack{123\\123}} & A_{\substack{123\\124}} & A_{\substack{123\\125}} & A_{\substack{123\\134}} & A_{\substack{123\\145}} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ A_{\substack{145\\123}} & A_{\substack{145\\124}} & A_{\substack{145\\125}} & A_{\substack{145\\134}} & A_{\substack{145\\145}} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} & a_{35} & 0 & 0 \\ a_{43} & a_{44} & a_{45} & 0 & 0 \\ a_{53} & a_{54} & a_{55} & 0 & 0 \\ 0 & a_{32} & 0 & a_{33} & -a_{35} \\ 0 & a_{52} & 0 & a_{53} & -a_{55} \end{vmatrix}$$

$$= \begin{vmatrix} A_{\substack{12\\12}} & 0 & 0 & -A_{\substack{12\\14}} & 0 \\ a_{34} & a_{44} & a_{54} & a_{32} & a_{52} \\ a_{35} & a_{45} & a_{55} & 0 & 0 \\ 0 & 0 & 0 & a_{33} & a_{53} \\ 0 & -A_{\substack{15\\12}} & A_{\substack{14\\12}} & 0 & -A_{\substack{14\\14}} \end{vmatrix}$$

$$= A_{\substack{12\\12}} \{ -a_{33}\, A_{\substack{14\\14}}\, A_{\substack{123\\123}} + a_{55}\, A_{\substack{15\\12}}\, A_{\substack{124\\145}} + a_{45}\, A_{\substack{14\\12}}\, A_{\substack{124\\145}} \}$$
$$- a_{53}\, A_{\substack{12\\14}} \{ A_{\substack{14\\12}}\, A_{\substack{125\\123}} - A_{\substack{15\\12}}\, A_{\substack{124\\123}} \}$$

$$= A_{\substack{12\\12}} \{ -a_{33}\, A_{\substack{14\\14}}\, A_{\substack{123\\123}} + (a_{55}\, A_{\substack{15\\12}} + a_{45}\, A_{\substack{14\\12}})\, A_{\substack{124\\145}} - a_{53}\, A_{\substack{12\\14}}\, A_{\substack{145\\123}} \},$$

which, after easy transformations,

$$= \underset{12}{A_{12}}\,\underset{124}{A_{124}}\{A_{11} - a_{44}\underset{14}{A_{14}} - a_{55}\underset{15}{A_{15}} - a_{45}\underset{15}{A_{14}} - a_{54}\underset{14}{A_{15}}\},$$

$$\therefore\quad (\underset{123}{A_{123}}\,\underset{145}{A_{145}}) = (a_{44}\underset{14}{A_{14}} + a_{55}\underset{15}{A_{15}} + a_{45}\underset{15}{A_{14}} + a_{54}\underset{14}{A_{15}} - A_{11}).$$

Similarly,

$$(\underset{124}{A_{124}}\,\underset{135}{A_{135}}) = (a_{33}\underset{13}{A_{13}} + a_{55}\underset{15}{A_{15}} + a_{35}\underset{15}{A_{13}} + a_{53}\underset{13}{A_{15}} - A_{11}),$$

$$\text{etc.,}\qquad\qquad\text{etc.}$$

Taking the sum of all principal minors of order two, we get

$$\Sigma\,(\underset{123}{A_{123}}\,\underset{124}{A_{124}}) = \Sigma a_{11}\Sigma\underset{12}{A_{12}} - \Sigma A_{11}.$$

14. In a similar manner it may be found that

$$\Sigma\,(\underset{123}{A_{123}}\,\underset{124}{A_{124}}\,\underset{125}{A_{125}}) = (\Sigma\underset{12}{A_{12}})^2 + (\Sigma a_{11})^2\Sigma A_{11} - 2\Sigma\underset{123}{A_{123}}\Sigma A_{11} - \Delta\Sigma a_{11},$$

$$\Sigma\,(\underset{123}{A_{123}}\,\underset{124}{A_{124}}\,\underset{125}{A_{125}}\,\underset{234}{A_{234}}) = (\Sigma a_{11})^3.\Delta + \Sigma a_{11}.\Sigma\underset{12}{A_{12}}.\Sigma A_{11} - 3\Sigma a_{11}.\Sigma\underset{123}{A_{123}}.\Delta - (\Sigma A_{11})^2 + \Sigma\underset{12}{A_{12}}.\Delta,$$

$$\text{etc.,}\qquad\qquad\text{etc.}$$

15. To find expressions for the principal minors of $\Delta_{(3)}$ of order two, we have the product

$$\begin{vmatrix} \underset{12}{A_{12}} & \underset{31}{A_{12}} & \underset{14}{A_{12}} & \underset{51}{A_{12}} \\ \underset{21}{A_{13}} & \underset{13}{A_{13}} & \underset{41}{A_{13}} & \underset{15}{A_{13}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} a_{22} & -a_{23} & a_{24} & -a_{25} \\ a_{32} & -a_{33} & a_{34} & -a_{35} \\ a_{42} & -a_{43} & a_{44} & -a_{45} \\ a_{52} & -a_{53} & a_{54} & -a_{55} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & 0 & 0 & 0 \\ 0 & -A_{11} & 0 & 0 \\ a_{24} & a_{34} & a_{44} & a_{54} \\ -a_{25} & -a_{35} & -a_{45} & -a_{55} \end{vmatrix}$$

$$= A_{11}^2\,\underset{123}{A_{123}},$$

$$\therefore\quad (\underset{12}{A_{12}}\,\underset{13}{A_{13}}) = A_{11}\,\underset{123}{A_{123}}.$$

Similarly,

$$(\underset{12}{A_{12}}\,\underset{14}{A_{14}}) = A_{11}\,\underset{124}{A_{124}},$$

$$(\underset{12}{A_{12}}\,\underset{15}{A_{15}}) = A_{11}\,\underset{125}{A_{125}},$$

$$\text{etc.,}\qquad\qquad\text{etc.}$$

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\31}} & A_{\substack{12\\14}} & A_{\substack{12\\51}} & A_{\substack{12\\42}} & A_{\substack{12\\34}} & A_{\substack{12\\45}} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ A_{\substack{45\\12}} & A_{\substack{45\\31}} & A_{\substack{45\\14}} & A_{\substack{45\\51}} & A_{\substack{45\\42}} & A_{\substack{45\\34}} & A_{\substack{45\\45}} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & -a_{23} & a_{24} & -a_{25} & 0 & 0 & 0 \\ a_{32} & -a_{33} & a_{34} & -a_{35} & 0 & 0 & 0 \\ a_{42} & -a_{43} & a_{44} & -a_{45} & 0 & 0 & 0 \\ a_{52} & -a_{53} & a_{54} & -a_{55} & 0 & 0 & 0 \\ 0 & 0 & a_{21} & 0 & -a_{22} & a_{23} & -a_{23} \\ 0 & 0 & a_{31} & 0 & -a_{32} & a_{33} & -a_{35} \\ 0 & 0 & a_{51} & 0 & -a_{52} & a_{53} & -a_{55} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & 0 & 0 & 0 & -A_{14} & 0 & 0 \\ -a_{23} & -a_{33} & -a_{43} & -a_{53} & 0 & 0 & 0 \\ a_{24} & a_{34} & a_{44} & a_{54} & a_{21} & a_{31} & a_{51} \\ -a_{25} & -a_{35} & -a_{45} & a_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{22} & a_{32} & a_{52} \\ 0 & 0 & 0 & 0 & a_{23} & a_{33} & a_{53} \\ 0 & 0 & -A_{51} & A_{41} & 0 & 0 & -A_{44} \end{vmatrix}$$

$$= A_{11}\, A_{\substack{14\\14}} \{ A_{44}\, A_{\substack{125\\125}} + A_{55}\, A_{\substack{124\\124}} - A_{45}\, A_{\substack{125\\124}} - A_{54}\, A_{\substack{124\\125}} - a_{33}\Delta \},$$

$$\therefore \quad (A_{\substack{12\\12}}\, A_{\substack{45\\45}}) = A_{44}\, A_{\substack{125\\125}} + A_{55}\, A_{\substack{124\\124}} - A_{45}\, A_{\substack{125\\124}} - A_{54}\, A_{\substack{124\\125}} - a_{33}\Delta.$$

Similarly,

$$(A_{\substack{12\\12}}\, A_{\substack{34\\34}}) = A_{44}\, A_{\substack{123\\123}} + A_{33}\, A_{\substack{124\\124}} - A_{43}\, A_{\substack{123\\124}} - A_{34}\, A_{\substack{124\\123}} - a_{55}\Delta,$$

etc., etc.

The sum of the last fifteen minors may be easily found to be

$$= A_{11}\,(A_{\substack{345\\345}} + A_{\substack{245\\245}} + A_{\substack{235\\235}} + A_{\substack{234\\234}}) + A_{22}\, A_{\substack{345\\345}} - \Delta\Sigma a_{11},$$

and therefore

$$\Sigma\,(A_{\substack{12\\12}}\, A_{\substack{13\\13}}) = \Sigma A_{11}\, \Sigma A_{\substack{123\\123}} - \Delta\Sigma a_{11}.$$

16. In a similar manner we may find that

$$\begin{aligned} \Sigma\,(A_{\substack{12\\12}}\, A_{\substack{13\\13}}\, A_{\substack{14\\14}}) &= \Sigma a_{11}\,(\Sigma A_{11})^2 \\ &\quad + \Delta\,\{(\Sigma A_{\substack{123\\123}})^2 - 2\Sigma a_{11}\, \Sigma A_{\substack{12\\12}}\} \\ &\quad - \Delta . \Sigma A_{11}, \\ \Sigma\,(A_{\substack{12\\12}}\, A_{\substack{13\\13}}\, A_{\substack{14\\14}}\, A_{\substack{15\\15}}) &= \Delta . \Sigma a_{11} . \Sigma A_{11} . \Sigma A_{\substack{123\\123}} + (\Sigma A_{11})^3 \\ &\quad - \Delta^2 . (\Sigma a_{11})^2 - 3\Delta . \Sigma A_{11} . \Sigma A_{\substack{12\\12}} \\ &\quad + \Delta^2 . \Sigma A_{\substack{123\\123}}, \end{aligned}$$

etc., etc.

C.—*Determinant of order* ω.

17. In the general case (for a determinant Δ of order ω) I shall adopt, for sake of convenience, the following notation:

$\Delta_{(m)}$ will denote the compound determinant formed from the minors of Δ of order m,

$\Delta_{(m)(n)}$ will denote the compound determinant formed from the minors of $\Delta_{(m)}$ of order n,

Δ_m will denote a minor of Δ of order m,

$\underset{rs}{\Delta_m}$ will denote the constituent in the r^{th} row and s^{th} column of $\Delta_{(m)}$,

$\Delta_{(m)\,n}$ will denote a minor of $\Delta_{(m)}$ of order n,

$\underset{rs}{\Delta_{(m)\,n}}$ will denote the constituent in the r^{th} row and s^{th} column of $\Delta_{(m)(n)}$;

$\Sigma\underset{ss}{\Delta_m}$ will denote the sum of the principal minors of Δ of order m,

$\Sigma\underset{ss}{\Delta_{(m)\,n}}$ will denote the sum of the principal minors of $\Delta_{(m)}$ of order n.

Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \ldots\ldots & a_{1\omega} \\ a_{21} & a_{22} & a_{23} \ldots\ldots & a_{2\omega} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{\omega 1} & a_{\omega 2} & a_{\omega 3} \ldots\ldots & a_{\omega\omega} \end{vmatrix},$$

then

$$\Delta_{(2)} = \begin{vmatrix} \underset{11}{\Delta_2} & \underset{12}{\Delta_2} & \underset{13}{\Delta_2} \ldots\ldots & \underset{1\omega_2}{\Delta_2} \\ \underset{21}{\Delta_2} & \underset{22}{\Delta_2} & \underset{23}{\Delta_2} \ldots\ldots & \underset{2\omega_2}{\Delta_2} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \underset{\omega_2 1}{\Delta_2} & \underset{\omega_2 2}{\Delta_2} & \underset{\omega_2 3}{\Delta_2} \ldots\ldots & \underset{\omega_2\omega_2}{\Delta_2} \end{vmatrix},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Delta_{(m)} = \begin{vmatrix} \underset{11}{\Delta_m} & \underset{12}{\Delta_m} & \underset{13}{\Delta_m} \ldots\ldots & \underset{1\omega_m}{\Delta_m} \\ \underset{21}{\Delta_m} & \underset{22}{\Delta_m} & \underset{23}{\Delta_m} \ldots\ldots & \underset{2\omega_m}{\Delta_m} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \underset{\omega_m 1}{\Delta_m} & \underset{\omega_m 2}{\Delta_m} & \underset{\omega_m 3}{\Delta_m} \ldots\ldots & \underset{\omega_m\omega_m}{\Delta_m} \end{vmatrix},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$
\Delta_{(\omega-1)} = \begin{vmatrix}
\underset{11}{\Delta_{\omega-1}} & \underset{12}{\Delta_{\omega-1}} & \underset{13}{\Delta_{\omega-1}} & \cdots & \underset{1\omega}{\Delta^*_{\omega-1}} \\
\underset{21}{\Delta_{\omega-1}} & \underset{22}{\Delta_{\omega-1}} & \underset{23}{\Delta_{\omega-1}} & \cdots & \underset{2\omega}{\Delta^*_{\omega-1}} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\underset{\omega 1}{\Delta^*_{\omega-1}} & \underset{\omega 2}{\Delta^*_{\omega-1}} & \underset{\omega 3}{\Delta^*_{\omega-1}} & \cdots & \underset{\omega\omega}{\Delta^*_{\omega-1}}
\end{vmatrix}.
$$

18. On observing that $\omega_m = \omega_{\omega-m}$, or more generally $(\omega-k)_m = (\omega-k)_{\omega-m-k}$, and that any minor of order p of the determinant $\Delta_{(\omega-m)}$ is equal to the complementary of the corresponding minor of $\Delta_{(m)}$ multiplied by $\Delta^{p-(\omega-1)_{m-1}}$,† we may write

$$
\begin{aligned}
\Sigma\underset{ss}{\Delta_{(2)\,\omega_2-1}} &= \Delta^{\omega-2}.\,\Sigma\underset{ss}{\Delta_{\omega-}}\ , \\
\Sigma\underset{ss}{\Delta_{(m)\,\omega_m-1}} &= \Delta^{(\omega-1)_{m-}\ -1}.\,\Sigma\underset{ss}{\Delta_{\omega-m}}, \\
\Sigma\underset{ss}{\Delta_{(\omega-m)\,\omega_m-1}} &= \Delta^{(\omega-1)_{\omega-m-1}-1}.\,\Sigma\underset{ss}{\Delta_m} &&= \Delta^{(\omega-1)_m-1}.\,\Sigma\underset{ss}{\Delta_m}, \\
\Sigma\underset{ss}{\Delta_{(\omega-m)\,\omega_m-n}} &= \Delta^{(\omega-1)_{\omega-m-1}-n}.\,\Sigma\underset{ss}{\Delta_{(m)\,n}} &&= \Delta^{(\omega-1)_m-n}.\,\Sigma\underset{ss}{\Delta_{(m)\,n}}, \\
\Sigma\underset{ss}{\Delta_{(\omega-1)\,\omega-1}} &= \Delta^{(\omega-1)_{\omega-2}-1}.\,\Sigma a_{11} &&= \Delta^{\omega-2}.\,\Sigma a_{11}, \\
\text{etc.,} & && \text{etc.}
\end{aligned}
$$

19. Proceeding in the same way as in the cases of determinants of the fourth and fifth order, we get

$$
\begin{aligned}
\Sigma\underset{ss}{\Delta_{(2)\,2}} &= \Sigma a_{ss}\Sigma\underset{ss}{\Delta_3} - \Sigma\underset{ss}{\Delta_4}, \\
\Sigma\underset{ss}{\Delta_{(3)\,2}} &= \Sigma\underset{ss}{\Delta_2}.\Sigma\underset{ss}{\Delta_4} - \Sigma a_{ss}\Sigma\underset{ss}{\Delta_5} + \Sigma\underset{ss}{\Delta_6}, \\
\Sigma\underset{ss}{\Delta_{(4)\,2}} &= \Sigma\underset{ss}{\Delta_3}\Sigma\underset{ss}{\Delta_5} - \Sigma\underset{ss}{\Delta_2}\Sigma\underset{ss}{\Delta_6} + \Sigma a_{ss}\Sigma\underset{ss}{\Delta_7} - \Sigma\underset{ss}{\Delta_8}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Sigma\underset{ss}{\Delta_{(m)\,2}} &= \Sigma\underset{ss}{\Delta_{m-1}}\Sigma\underset{ss}{\Delta_{m+1}} - \Sigma\underset{ss}{\Delta_{m-2}}\Sigma\underset{ss}{\Delta_{m+2}} + \cdots \\
&\qquad + (-1)^{m-2}\Sigma a_{ss}\Sigma\underset{ss}{\Delta_{2m-1}} + (-1)^{m-1}\Sigma\underset{ss}{\Delta_{2m}}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Sigma\underset{ss}{\Delta_{(\omega-2)\,2}} &= \Sigma\underset{ss}{\Delta_{\omega-3}}\Sigma\underset{ss}{\Delta_{\omega-1}} - \Sigma\underset{ss}{\Delta_{\omega-4}}.\,\Delta, \\
\Sigma\underset{ss}{\Delta_{(2)\,3}} &= (\Sigma\underset{ss}{\Delta_3})^2 + \Sigma\underset{ss}{\Delta_4}\{(\Sigma a_{ss})^2 - 2\Sigma\underset{ss}{\Delta_2}\} - \Sigma a_{ss}\Sigma\underset{ss}{\Delta_5} + \Sigma\underset{ss}{\Delta_6}, \\
\Sigma\underset{ss}{\Delta_{(3)\,3}} &= (\Sigma\underset{ss}{\Delta_2})^2\Sigma\underset{ss}{\Delta_5} + \Sigma a_{ss}(\Sigma\underset{ss}{\Delta_4})^2 - 2\Sigma a_{ss}\Sigma\underset{ss}{\Delta_3}\Sigma\underset{ss}{\Delta_5} - \Sigma a_{ss}\Sigma\underset{ss}{\Delta_2}\Sigma\underset{ss}{\Delta_6} \\
&\qquad + (\Sigma a_{ss})^2\Sigma\underset{ss}{\Delta_7} - \Sigma\underset{ss}{\Delta_4}\Sigma\underset{ss}{\Delta_5} + 3\Sigma\underset{ss}{\Delta_3}\Sigma\underset{ss}{\Delta_6} - \Sigma\underset{ss}{\Delta_2}\Sigma\underset{ss}{\Delta_7} - \Sigma a_{ss}\Sigma\underset{ss}{\Delta_8} + \Sigma\underset{ss}{\Delta_9}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\text{etc.,} & \qquad\qquad \text{etc.}
\end{aligned}
$$

* These constituents would be written $\underset{1\omega_{\omega-1}}{\Delta_{\omega-1}}$, $\underset{2\omega_{\omega-1}}{\Delta_{\omega-1}}$, etc., but since $\omega_{\omega-1} = \omega$, we may write them as above.

† *Vide* Scott, Chap. V, Art. 10.

Having found $\Sigma\Delta_{\substack{(m)\,n\\ ss}}$ for values of $m \overline{\gtrless} \frac{\omega}{2}$, we may readily obtain $\Sigma\Delta_{\substack{(m)\,n\\ ss}}$ for values of $m > \frac{\omega}{2}$ by applying what Muir terms the *Law of Complementaries.**

Applying these expressions to the relations given in Art. 18, we have

$$\Sigma\Delta_{\substack{(\omega-2)\,\omega_2-2\\ ss}} = \Delta^{(\omega-1)_2-2}\,(\Sigma a_{ss}\Sigma\Delta_{\substack{3\\ ss}} - \Sigma\Delta_{\substack{4\\ ss}}),$$

or generally

$$\Sigma\Delta_{\substack{(\omega-m)\,\omega_m-2\\ ss}} = \Delta^{(\omega-1)_m-2}\,(\Sigma\Delta_{\substack{m-1\\ ss}}\Sigma\Delta_{\substack{m+1\\ ss}}\cdot\ldots + (-1)^{m-2}\Sigma a_{ss}\Sigma\Delta_{\substack{2m-1\\ ss}} + (-1)^{m-1}\Sigma\Delta_{\substack{2m\\ ss}}).$$

§3.—*Roots of the Equation* $|\Delta_{m)} - x| = 0$.

20. Let us for convenience use $|\Delta_{(m)} - x|$ to denote the determinant $\Delta_{(m)}$ with the quantity x subtracted from each constituent along the principal diagonal. Then $|\Delta_{(m)} - x| = 0$ will be a polynomial in x of the order ω_m.

(a). If Δ is of the third order, we have the system of equations:

$$(-1)^3.|\Delta \;\; - x| = x^3 - \Sigma a_{ss}\,x_2 + \Sigma\Delta_{\substack{2\\ ss}}x \;\;\; - \Delta \;= 0, \tag{1}$$

$$(-1)^3.|\Delta_{(2)} - x| = x^3 - \Sigma\Delta_{\substack{2\\ ss}}x^2 + \Sigma\Delta_{\substack{(2)\,2\\ ss}}x - \Delta^2 = 0, \tag{2}$$

which may, in virtue of the relations found in Art. 19, be written

$$x^3 - \Sigma a_{ss}x^2 + \Sigma\Delta_{\substack{2\\ ss}}x \;\;\; - \Delta \;= 0, \tag{1}$$

$$x^3 - \Sigma\Delta_{\substack{2\\ ss}}x^2 + \Delta\Sigma a_{ss}\,x - \Delta^2 = 0. \tag{2}$$

From these equations we see that if the roots of equation (1) are g_1, g_2, g_3, the roots of (2) are g_1g_2, g_2g_3, g_1g_3. That is, the roots of equation (2) are the products of the roots of equation (1) taken two at a time.

(b). If Δ is of the fourth order, we have the system of equations

$$(-1)^4|\Delta - x| \;\; = x^4 - \Sigma a_{ss}x^3 + \Sigma\Delta_{\substack{2\\ ss}}x^2 - \Sigma\Delta_{\substack{3\\ ss}}x + \Delta = 0, \tag{1}$$

$$\begin{aligned}(-1)^6|\Delta_{(2)} - x| = x^6 - \Sigma\Delta_{\substack{2\\ ss}}x^5 + \Sigma\Delta_{\substack{(2)\,2\\ ss}}x^4 - \Sigma\Delta_{\substack{(2)\,3\\ ss}}x^3 + \Sigma\Delta_{\substack{(2)\,4\\ ss}}x^2 \\ - \Sigma\Delta_{\substack{(2)\,5\\ ss}}x + \Delta^3 = 0,\end{aligned} \tag{2}$$

$$(-1)^4|\Delta_{(3)} - x| = x^4 - \Sigma\Delta_{\substack{3\\ ss}}x^3 + \Sigma\Delta_{\substack{(3)\,2\\ ss}}x^2 - \Sigma\Delta_{\substack{(3)\,3\\ ss}}x + \Delta^2 = 0. \tag{3}$$

---

* Muir's Theory of Determinants, Chap. III, §98.

On substituting the values already found for the coefficients in these equations, we may write them as follows:

$$x^4 - \Sigma a_{ss} x^3 + \Sigma \underset{ss}{\Delta_2} x^2 - \Sigma \underset{ss}{\Delta_3} x + \Delta = 0, \tag{1}$$

$$x^6 - \Sigma \underset{ss}{\Delta_2} x^5 + (\Sigma a_{ss} \Sigma \underset{ss}{\Delta_3} - \Delta)\, x^4 - \{(\Sigma \underset{ss}{\Delta_3})^2 + \Delta(\Sigma a_{ss})^2 - 2\Sigma \underset{ss}{\Delta_2}\} x^3$$
$$+ \Delta(\Sigma a_{ss} \Sigma \underset{ss}{\Delta_3} - \Delta)\, x^2 - \Delta^2 \Sigma \underset{ss}{\Delta_2} x + \Delta^3 = 0, \tag{2}$$

$$x^4 - \Sigma \underset{ss}{\Delta_3} x^3 + \Delta \Sigma \underset{ss}{\Delta_2} x^2 - \Delta^2 \Sigma a_{ss} x + \Delta^3 = 0. \tag{3}$$

The coefficients in these equations show us that if g_1, g_2, g_3, g_4 are the roots of equation (1), then $g_1g_2, g_1g_3, g_1g_4, g_2g_3, g_2g_4, g_3g_4$ are the roots of equation (2), and $g_1g_2g_3, g_1g_2g_4, g_1g_3g_4, g_2g_3g_4$, are the roots of equation (3). That is, the roots of equation (2) are the products, two at a time, of the roots of equation (1), and the roots of equation (3) are the products, three at a time, of the roots of equation (1).

(c). It is apparently true in general that if Δ is a determinant of order ω, and if the roots of the equation

$$(-1)^{\omega} |\Delta - x| = 0 \tag{1}$$

are $g_1, g_2, g_3, \ldots . g_\omega$, then the roots of the equation

$$(-1)^{\omega_2} |\Delta_{(2)} - x| = 0 \tag{2}$$

are the products of the roots of equation (1) taken two at a time,

. . . .

the roots of the equation

$$(-1)^{\omega_m} |\Delta_{(m)} - x| = 0 \tag{m}$$

are the products of the roots of equation (1) taken m at a time,

etc., etc.

§4.—*Symmetric and Skew Symmetric Determinants.*

21. If a symmetric determinant Δ of order ω vanishes, and if all principal minors of order $\omega - 1$ vanish, then all minors of order $\omega - 1$ vanish.

That is, if $\Delta = 0$,

and $\underset{ss}{\Delta_{\omega-1}} = 0$ for all values of s,

then $\underset{rs}{\Delta_{\omega-1}} = 0$ " " " r and s.

Proof: The determinant $\Delta_{(\omega-1)}$ is symmetric (Art. 3), and therefore $\Delta_{\underset{ij}{\omega-1}} = \Delta_{\underset{ji}{\omega-1}}$; and since $\Delta_{\underset{ss}{(\omega-1)2}}$ represents any principal minor of $\Delta_{(\omega-1)}$ of order two it may be written

$$\Delta_{\underset{ss}{(\omega-1)2}} = \Delta_{\underset{ii}{\omega-1}}\Delta_{\underset{jj}{\omega-1}} - \Delta_{\underset{ij}{\omega-1}}\Delta_{\underset{ji}{\omega-1}} = -(\Delta_{\underset{ij}{\omega-1}})^2, \text{ since } \Delta_{\underset{ss}{\omega-1}} = 0.$$

But $\Delta_{\underset{ss}{(\omega-1)2}} = 0$ (Art. 5),

$$\therefore \quad \Delta_{\underset{ij}{\omega-1}} = 0 \text{ for all values of } i \text{ and } j;$$

that is, all minors of order $\omega - 1$ vanish.

22. We may now prove the following more general theorem:

If a symmetric determinant Δ of order ω vanishes, and if the sums of all principal minors of orders $\omega-1, \omega-2, \omega-3, \ldots \omega-m+1$ respectively vanish, and if all principal minors of order $\omega-m$ vanish, then all minors of order $\omega-m$ vanish.

That is, if

$$\begin{aligned} \Delta &= 0, \\ \Sigma\Delta_{\underset{ss}{\omega-1}} &= 0, \\ \Sigma\Delta_{\underset{ss}{\omega-2}} &= 0, \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ \Sigma\Delta_{\underset{ss}{\omega-m+1}} &= 0, \end{aligned}$$

and if $\quad \Delta_{\underset{ss}{\omega-m}} = 0$ for all values of s,

then $\quad \Delta_{\underset{rs}{\omega-m}} = 0$ " " " r and s.

Proof: Let

$$\Delta_{\underset{ss}{(\omega-m)2}} = \Delta_{\underset{ii}{\omega-m}}\Delta_{\underset{jj}{\omega-m}} - \Delta_{\underset{ij}{\omega-m}}\Delta_{\underset{ji}{\omega-m}},$$

and since $\Delta_{(\omega-m)}$ is symmetric and $\Delta_{\underset{ss}{\omega-m}} = 0$ for all values of s, we have

$$\Delta_{\underset{ss}{(\omega-m)2}} = -(\Delta_{\underset{ij}{\omega-m}})^2,$$

and $\quad \therefore \quad \Sigma\Delta_{\underset{ss}{(\omega-m)2}} = -\Sigma(\Delta_{\underset{ij}{\omega-m}})^2.$

But

$$\Sigma\Delta_{\underset{ss}{(\omega-m)2}} = \Sigma\Delta_{\underset{ss}{\omega-m-1}}\Sigma\Delta_{\underset{ss}{\omega-m+1}} - \Sigma\Delta_{\underset{ss}{\omega-m-2}}\Sigma\Delta_{\underset{ss}{\omega-m+2}} + \ldots + (-1)^{\omega-m-1}\Sigma\Delta_{\underset{ss}{\omega-2m}}\cdot\Delta = 0 \text{ (Art. 19)},$$

since one factor of every term is zero, and

$$\therefore \quad \Sigma\,(\Delta_{\underset{ij}{\omega - m}})^2 = 0,$$

consequently $$\Delta_{\underset{ij}{\omega - m}} = 0 \text{ for all values of } i \text{ and } j;$$

that is, all minors of order $\omega - m$ vanish.

23. This theorem is also true for skew symmetric determinants, for if Δ is skew symmetric and if

$$\Delta_{\underset{ss}{(\omega - m)2}} = \Delta_{\underset{ii}{\omega - m}}\Delta_{\underset{jj}{\omega - m}} - \Delta_{\underset{ij}{\omega - m}}\Delta_{\underset{ji}{\omega - m}},$$

then since $\Delta_{(\omega - m)}$ is either symmetric or skew symmetric according as $\omega - m$ is even or odd, we get, as in the case of Δ symmetric, that

$$\Sigma\,(\Delta_{\underset{ij}{\omega - m}})^2 = 0,$$

and $$\therefore \quad \Delta_{\underset{ij}{\omega - m}} = 0 \text{ for all values of } i \text{ and } j.$$

It is to be observed, however, that the theorem for skew symmetric determinants may be stated as follows:

If a skew symmetric determinant Δ of order ω vanishes, and if the sums of all principal minors of orders $\omega - 1$, $\omega - 2$, ... $\omega - m$ respectively vanish, then all minors of order $\omega - m$ vanish.

That is, if
$$\Delta = 0,$$
$$\Sigma\Delta_{\underset{ss}{\omega - 1}} = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
and $$\Sigma\Delta_{\underset{ss}{\omega - m}} = 0,$$
then $$\Delta_{\underset{rs}{\omega - m}} = 0 \text{ for all values of } r \text{ and } s.$$

For since a skew symmetric determinant of even order is a perfect square and a skew symmetric determinant of odd order vanishes, and since the principal minors of Δ are skew symmetric determinants of either odd or even order, it follows from the statement that the sum of all principal minors of order $\omega - m$ vanishes, that every principal minor of order $\omega - m$ vanishes.

§5.—*Determinants in General.*

24. If Δ is any determinant of order ω which vanishes, and if all principal minors of order $\omega-1$ vanish, then at least $\omega^2-\omega+1$ of the minors of order $\omega-1$ vanish.

For if
$$\Delta = 0,$$
and
$$\Delta_{\substack{\omega-1\\ ss}} = 0,$$
then
$$\Delta_{\substack{(\omega-1)2\\ rs}} = 0,$$
and
$$\Delta_{(\omega-1)} = \left|\begin{array}{cccc} 0 & \Delta_{\substack{\omega-1\\ 12}} & \cdots\cdots & \Delta_{\substack{\omega-1\\ 1\omega}} \\ \Delta_{\substack{\omega-1\\ 21}} & 0 & \cdots\cdots & \Delta_{\substack{\omega-1\\ 2\omega}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_{\substack{\omega-1\\ \omega1}} & \Delta_{\substack{\omega-1\\ \omega2}} & \cdots\cdots & 0 \end{array}\right|.$$

Now
$$\Delta_{\substack{\omega-1\\ 21}} \cdot \Delta_{\substack{\omega-1\\ s2}} = 0, \qquad (s = 1, 3, 4, \ldots . \omega),$$
therefore if
$$\Delta_{\substack{\omega-1\\ 21}} \neq 0,$$
then
$$\Delta_{\substack{\omega-1\\ 12}} = \Delta_{\substack{\omega-1\\ 32}} = \ldots . = \Delta_{\substack{\omega-1\\ \omega2}} = 0.$$

Again,
$$\Delta_{\substack{\omega-1\\ 31}} \cdot \Delta_{\substack{\omega-1\\ s3}} = 0, \qquad (s = 1, 2, 4, \ldots . \omega),$$
and if either of these factors does not vanish, then there are $\omega-1$ others (besides those already zero) which do vanish. In this way we show that for every constituent which does not vanish there are $\omega-1$ which do. There cannot, therefore, be more than $\omega-1$ constituents which do not vanish, or in other words, there are at least $\omega^2-(\omega-1) = \omega^2-\omega+1$ which do.

II.—MATRICES.

25. As an immediate consequence of Art. 23 we have the following theorem in matrices:

Every skew symmetric matrix has a nullity equal to its vacuity.

26. If ϕ is an orthogonal matrix of order ω, and if $\phi_{(m)}$ denote the matrix whose constituents are the minors of order m of the content of ϕ, then each of the matrices $\phi_{(2)}$, $\phi_{(3)} \ldots . \phi_{(\omega-1)}$ is orthogonal.

This follows from the fact that every minor of $|\phi|$ is equal to its complementary,* and therefore

$$\phi_{(m)} = \phi_{(\omega - m)} .$$

We know that
$$\phi_{(m)} \breve{\phi}_{(\omega - m)} = 1 ,$$
$$\therefore \quad \phi_{(m)} \breve{\phi}_{(m)} = 1 .$$

27. If ϕ and ψ are two orthogonal matrices of the same odd order, then the matrix $(\phi \dot{-} \psi)$ is vacuous.

This follows from the theorem that if ϕ_{ik} denote the constituent of ϕ in the i^{th} row and k^{th} column, then the determinant $|\lambda\phi_{ik} + \mu\psi_{ik}|$ is not altered by interchanging λ and μ.†

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* *Vide* Scott, Chap. XI, Art. 16.

† *Vide* Scott, Chap. XI, Art. 17.